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Gauge theories on the κ-Minkowski spacetime

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Abstract. This study of gauge field theories on κ-deformed Minkowski spacetime extends previous work on field theories on this example of a non-commutative spacetime. We construct deformed gauge theories for arbitrary compact Lie groups using the concept of enveloping algebra-valued gauge transformations and the Seiberg–Witten formalism. Derivative-valued gauge fields lead to field strength tensors as the sum of curvature- and torsion-like terms. We construct the Lagrangians explicitly to first order in the deformation parameter. This is the first example of a gauge theory that possesses a deformed Lorentz covariance.

1 Introduction

The best known description of the fundamental forces of nature is given by gauge theories. Nevertheless intrinsic difficulties arise in these theories at very high energies or very short distances. Physics is not very well known in this limit. This has lead to the idea of modifying the structure of spacetime at very short distances and to introduce uncertainty relations for the coordinates to provide a natural cut-off (for reviews of this wide field see [1,2]). It is expected that gauge theories still play a vital role in this regime.

We expect especially interesting new features of gauge field theories formulated on spaces with a deformed spacetime symmetry. Here we concentrate on the κ-deformed Poincaré algebra (introduced in [3–5][1]). The spacetime which is covariant with respect to this deformed symmetry algebra is called the κ-deformed quantum space.

In a previous paper [6] deformed field theories on a κ-deformed quantum space have been constructed. The techniques necessary for such a construction have been thoroughly discussed there. In this paper we show how the deformation concept can be applied to a gauge field theory. We construct deformed gauge theories for arbitrary compact Lie groups. "Deformed" does not mean that the Lie groups will be deformed, however, the transformation parameters will depend on the elements of the κ-deformed coordinate space. This implies that Lie algebra gauge transformations are generalized to enveloping algebra-valued gauge transformations.

This is possible by making use of the Seiberg–Witten map [7–9]. This is a map that allows one to express all elements of the non-commutative gauge theory by their commutative analogs. It follows that a deformed gauge theory can be constructed with exactly the same number of fields (gauge fields, matter fields) as the standard gauge theory on undeformed space.

Of special interest is the interplay of the gauge transformations with the κ-deformed Lorentz transformations. Gauge theories are based on the concept of covariant derivatives constructed with gauge fields. Covariance now refers both to the gauge transformations and to the κ-Lorentz transformations.

Theories like the one presented here can be used to deform the standard model (compare with the approach in [10, 11]). For example, new coupling terms in the Lagrangian arise. This has experimental consequences and the model can be tested phenomenologically. We exhibit these terms for an arbitrary gauge group to first order in the deformation parameter. These models should be understood in such an expansion; it renders an infrared cutoff. We do not assume that these models should be used to describe physics at large distances.

To obtain phenomenologically interesting results we develop the theory on a space-like κ-deformed spacetime with Minkowski signature (in [6] κ-deformed Euclidean spacetime was discussed).

This paper is organized as follows: In the first section we present a compilation of all relevant formulae for κ-Minkowski spacetime. To derive and understand these formulae, [6] is essential. In the second section we investigate covariant derivatives for enveloping algebra-valued gauge transformations. Attention is given to the κ-Lorentz covariance as well as to gauge covariance. For this purpose the enveloping algebra-valued gauge formalism is devel-

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[1] For additional references concerning this model see [6].

oped and the transformation property of the gauge field is derived. This leads to the new concept of derivative-valued gauge fields. The field strength tensors, defined by commutators of covariant derivatives, are derivative dependent as well. We expand them in terms of covariant derivatives and show that each expansion coefficient is a tensor by itself; we call them torsion-like tensors. The derivative-independent term is a deformation of $F^0_{\mu\nu}$ and is used in the construction of Lagrangians.

In the third section we construct the Seiberg–Witten map. We use the $\star$-product formalism and expand in the deformation parameter. All gauge and matter fields of the deformed theory can be expressed in terms of the standard Lie algebra gauge fields and the standard matter fields. This allows for the construction of a Lagrangian in terms of the standard fields.

In the fourth section we discuss the interplay of κ-Lorentz and gauge transformations. We show that gauge transformations and κ-Lorentz transformations commute and that the κ-Lorentz transformed gauge transformation reproduces again the algebra. This can be implemented in a more abstract setting, but we discuss this issue explicitly in order to become familiar with the comultiplication rules and their consequences.

2 The κ-Minkowski space

In a previous paper we discussed the κ-Euclidean space [6] (introduced in [3,4]) and argued that the generalization to a Minkowski version is straightforward. We introduce here this space-like κ-deformed Minkowski spacetime, which is more interesting for phenomenological applications. First we present the relevant formulae.

Coordinate space

We start from the same algebra as in [6]:

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}(a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu), \qquad \mu, \nu = 0, 1, \ldots, n, \tag{1}$$

but the metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$ and its inverse are used to raise and lower indices. Space-like deformation will be achieved by assuming a^μ to be space-like. The n-direction is rotated in the direction of a^μ, $a^n = a$, $a^j = 0$. We label the n commuting coordinates with $\hat{x}^i$ ($i = 0, \ldots, n-1$) as opposed to $\hat{x}^n$ and obtain the following commutation relations:

$$[\hat{x}^n, \hat{x}^j] = \mathrm{i}a\hat{x}^j, \quad [\hat{x}^i, \hat{x}^j] = 0, \quad i, j = 0, 1, \ldots, n-1. \tag{2}$$

The parameter a is related to the frequently used parameter κ:

$$a = \frac{1}{\kappa}. \tag{3}$$

The κ-deformed Lorentz algebra

The formulae for the transformation of the spacetime coordinates are the same as for the Euclidean space, replacing $\delta^{\mu\nu}$ with $\eta^{\mu\nu}$ and paying attention to upper and lower indices:

$$\begin{aligned}
[M^{ij}, \hat{x}^\mu] &= \eta^{\mu j}\hat{x}^i - \eta^{\mu i}\hat{x}^j, \\
[M^{in}, \hat{x}^\mu] &= \eta^{\mu n}\hat{x}^i - \eta^{\mu i}\hat{x}^n + \mathrm{i}aM^{i\mu},
\end{aligned} \tag{4}$$

$\mu = 0, 1, \ldots, n$. These relations are consistent with the algebra (2) and they lead to the undeformed algebra relations of the generators $M^{\mu\nu}$ of the Lorentz algebra $so(1,n)$:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}. \tag{5}$$

As in [6], this is again the undeformed algebra [5]. However, the generators act in a deformed way on products of functions (i.e. they have a deformed coproduct)

$$\begin{aligned}
M^{ij}(\hat{f}\cdot\hat{g}) &= (M^{ij}\hat{f})\cdot\hat{g} + \hat{f}\cdot(M^{ij}\hat{g}), \\
M^{in}(\hat{f}\cdot\hat{g}) &= (M^{in}\hat{f})\cdot\hat{g} + (\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(M^{in}\hat{g}) \\
&\quad + \mathrm{i}a(\hat{\partial}_k\hat{f})\cdot(M^{ik}\hat{g}).
\end{aligned} \tag{6}$$

In this paper we adopt the convention that over double Latin indices should be summed from 0 to $n-1$ and over double Greek from 0 to n.

2.1 Derivatives

We introduce derivatives in such a way that they are consistent with (2):

$$\begin{aligned}
[\hat{\partial}_n, \hat{x}^\mu] &= \eta_n{}^\mu, \\
[\hat{\partial}_i, \hat{x}^\mu] &= \eta_i{}^\mu - \mathrm{i}a\eta^{\mu n}\hat{\partial}_i.
\end{aligned} \tag{7}$$

Derivatives naturally carry a lower index. It is possible to derive from (7) the Leibniz rule (i.e. the coproduct):

$$\begin{aligned}
\hat{\partial}_n\,(\hat{f}\cdot\hat{g}) &= (\hat{\partial}_n\hat{f})\cdot\hat{g} + \hat{f}\cdot(\hat{\partial}_n\hat{g}), \\
\hat{\partial}_i\,(\hat{f}\cdot\hat{g}) &= (\hat{\partial}_i\hat{f})\cdot\hat{g} + (\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(\hat{\partial}_i\hat{g}).
\end{aligned} \tag{8}$$

The derivatives are a κ-Lorentz algebra module:

$$\begin{aligned}
[M^{ij}, \hat{\partial}_\mu] &= \eta^j{}_\mu\hat{\partial}^i - \eta^i{}_\mu\hat{\partial}^j, \\
[M^{in}, \hat{\partial}_n] &= \hat{\partial}^i, \\
[M^{in}, \hat{\partial}_j] &= \eta^i{}_j\frac{\mathrm{e}^{2\mathrm{i}a\hat{\partial}_n} - 1}{2\mathrm{i}a} - \frac{\mathrm{i}a}{2}\eta^i{}_j\hat{\partial}^l\hat{\partial}_l + \mathrm{i}a\hat{\partial}^i\hat{\partial}_j.
\end{aligned} \tag{9}$$

The part of $M^{\mu\nu}$ that acts on the coordinates and derivatives can be expressed in terms of the coordinates and the derivatives:

$$\begin{aligned}
\hat{M}^{ij} &= \hat{x}^i\hat{\partial}^j - \hat{x}^j\hat{\partial}^i, \\
\hat{M}^{in} &= \hat{x}^i\frac{1 - \mathrm{e}^{2\mathrm{i}a\hat{\partial}_n}}{2\mathrm{i}a} - \hat{x}^n\hat{\partial}^i + \frac{\mathrm{i}a}{2}\hat{x}^i\hat{\partial}^l\hat{\partial}_l.
\end{aligned} \tag{10}$$

Dirac operator

The κ-deformed Dirac operator has the components:

$$\hat{D}_n = \frac{1}{a}\sin(a\hat{\partial}_n) - \frac{\mathrm{i}a}{2}\hat{\partial}^l\hat{\partial}_l \mathrm{e}^{-\mathrm{i}a\hat{\partial}_n},$$
$$\hat{D}_i = \hat{\partial}_i \mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}, \tag{11}$$
$$[M^{\mu\nu}, \hat{D}_\rho] = \eta^\nu{}_\rho \hat{D}^\mu - \eta^\mu{}_\rho \hat{D}^\nu. \tag{12}$$

It can be seen as a derivative as well and satisfies the Leibniz rule:

$$\hat{D}_n(\hat{f}\cdot\hat{g}) = (\hat{D}_n\hat{f})\cdot(\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\hat{g}) + (\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(\hat{D}_n\hat{g}) -\mathrm{i}a(\hat{D}_i\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(\hat{D}^i\hat{g}), \tag{13}$$
$$\hat{D}_i\ (\hat{f}\cdot\hat{g}) = (\hat{D}_i\hat{f})\cdot(\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\hat{g}) + \hat{f}\cdot(\hat{D}_i\hat{g}).$$

That the Dirac operator really acts as a derivative follows from the commutation relations, when we expand the square root:

$$[\hat{D}_n, \hat{x}^j] = -\mathrm{i}a\hat{D}^j,$$
$$[\hat{D}_n, \hat{x}^n] = \sqrt{1 + a^2\hat{D}^\mu\hat{D}_\mu},$$
$$[\hat{D}_i, \hat{x}^j] = \eta_i{}^j\left(-\mathrm{i}a\hat{D}_n + \sqrt{1 + a^2\hat{D}^\mu\hat{D}_\mu}\right), \tag{14}$$
$$[\hat{D}_i, \hat{x}^n] = 0.$$

⋆-product

In the ⋆-product formulation (explained in detail in [6]) all the elements of the coordinate algebra can be realized as functions of commuting variables. Derivatives and the κ-Lorentz algebra generators can be realized in terms of commuting variables and derivatives. On the ⋆-product of functions they act with their comultiplication rules without seeing the x and derivative dependence of the ⋆-product.

Here we present for convenience a compilation of the relevant formulae used in the rest of this paper.

The κ-Minkowski spacetime (2) can be considered as a Lie algebra with $C^{\mu\nu}_\lambda = a(\eta^\mu{}_n\eta^\nu{}_\lambda - \eta^\nu{}_n\eta^\mu{}_\lambda)$ as structure constants. These structure constants appear also in the expansion of the symmetric ⋆-product:

$$\begin{aligned} f\star g(x) &= \lim_{\substack{y\to x\\ z\to x}} \exp\Bigg(\frac{\mathrm{i}}{2}x^\lambda C^{\mu\nu}_\lambda \partial_\mu\otimes\partial_\nu \\ &\quad -\frac{a}{12}x^\lambda C^{\mu\nu}_\lambda(\partial_n\partial_\mu\otimes\partial_\nu - \partial_\mu\otimes\partial_n\partial_\nu) + \dots\Bigg) \\ &\quad \times f(y)\otimes g(z) \\ &= f(x)g(x) + \frac{\mathrm{i}a}{2}x^j(\partial_n f(x)\partial_j g(x) - \partial_j f(x)\partial_n g(x)) \\ &\quad + \dots. \end{aligned} \tag{15}$$

The derivatives

$$\partial_n^* f(x) = \partial_n f(x),$$
$$\partial_i^* f(x) = \frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1}{\mathrm{i}a\partial_n}\partial_i f(x) \tag{16}$$

have the Leibniz rule

$$\begin{aligned} &\partial_n^*(f(x)\star g(x)) \\ &\quad = (\partial_n^* f(x))\star g(x) + f(x)\star(\partial_n^* g(x)), \\ &\partial_i^*(f(x)\star g(x)) \\ &\quad = (\partial_i^* f(x))\star g(x) + (\mathrm{e}^{\mathrm{i}a\partial_n^*} f(x))\star(\partial_i^* g(x)). \end{aligned} \tag{17}$$

The Dirac operator

$$D_n^* f(x) = \left(\frac{1}{a}\sin(a\partial_n) - \frac{\cos(a\partial_n) - 1}{\mathrm{i}a\partial_n^2}\partial_j\partial^j\right) f(x),$$
$$D_i^* f(x) = \frac{\mathrm{e}^{-\mathrm{i}a\partial_n} - 1}{-\mathrm{i}a\partial_n}\partial_i f(x) \tag{18}$$

has the following Leibniz rule:

$$\begin{aligned} &D_n^*(f(x)\star g(x)) \\ &\quad = (D_n^* f(x))\star(\mathrm{e}^{-\mathrm{i}a\partial_n^*}g(x)) + (\mathrm{e}^{\mathrm{i}a\partial_n^*}f(x))\star(D_n^* g(x)) \\ &\qquad -\mathrm{i}a\left(D_j^*\mathrm{e}^{\mathrm{i}a\partial_n^*}f(x)\right)\star(D^{j*}g(x)), \\ &D_i^*(f(x)\star g(x)) \\ &\quad = (D_i^* f(x))\star(\mathrm{e}^{-\mathrm{i}a\partial_n^*}g(x)) + f(x)\star(D_i^* g(x)). \end{aligned} \tag{19}$$

The generators of κ-Lorentz transformations, acting on coordinates and derivatives,

$$\begin{aligned} &M^{*in}f(x) \\ &\quad = \Bigg(x^i\partial^n - x^n\partial^i + x^i\partial_\mu\partial^\mu\frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1}{2\partial_n} \\ &\qquad + x^\mu\partial_\mu\partial^i\frac{a\partial_n + \mathrm{i}(\mathrm{e}^{\mathrm{i}a\partial_n} - 1)}{a\partial_n^2}\Bigg) f(x), \\ &M^{*ij}f(x) = \left(x^i\partial^j - x^j\partial^i\right) f(x), \end{aligned} \tag{20}$$

have the following coproduct:

$$\begin{aligned} &M^{*in}\left(f(x)\star g(x)\right) \\ &\quad = \left(M^{*in}f(x)\right)\star g(x) + \left(\mathrm{e}^{\mathrm{i}a\partial_n^*}f(x)\right)\star\left(M^{*in}g(x)\right) \\ &\qquad +\mathrm{i}a\left(\partial_j^* f(x)\right)\star\left(M^{*ij}g(x)\right), \\ &M^{*ij}\left(f(x)\star g(x)\right) \\ &\quad = (M^{*ij}f(x))\star g(x) + f(x)\star(M^{*ij}g(x)). \end{aligned} \tag{21}$$

Thus, the entire calculus developed in the abstract algebra can be formulated in the ⋆-product setting. For applications in physics this is of advantage because functions of commuting variables x are suitable representations of physical objects like fields.

3 Covariant derivatives

Gauge theories will be formulated with the help of covariant derivatives. We shall demand covariance under the κ-Lorentz algebra as well as covariance under the gauge group. Gauge fields have to be vector fields that transform like the Dirac operator under the deformed Lorentz algebra to render a covariant theory.

κ-Lorentz covariance

We start from the definition of a scalar field. In an undeformed theory this would be $\phi'(x') = \phi(x)$. For noncommuting variables we try however to avoid $\hat{x}'$. Note that

$$\hat{x}'^\mu \hat{x}'^\nu \neq (1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{x}^\mu \hat{x}^\nu. \quad (22)$$

Therefore we replace $\hat{\phi}'(\hat{x}')$ with $(1+\epsilon_{\mu\nu}\hat{M}^{\mu\nu})\phi'(\hat{x})$, where $\hat{M}^{\mu\nu}$ acts on the coordinates and the derivatives and has been defined in (10). The defining equation for a scalar field will take the form

$$(1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{\phi}'(\hat{x}) = \hat{\phi}(\hat{x}), \quad (23)$$

with the immediate consequence

$$\hat{\phi}'(\hat{x}) = \hat{\phi}(\hat{x}) - \epsilon_{\mu\nu}\hat{M}^{\mu\nu}\hat{\phi}(\hat{x}). \quad (24)$$

To compute the transformation law of a derivative of a scalar field we calculate $(1+\epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{D}_\rho\hat{\phi}'(\hat{x})$ that replaces $\hat{D}'_\rho\hat{\phi}'(\hat{x}')$:

$$\begin{aligned}&(1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{D}_\rho\hat{\phi}'(\hat{x}) \quad (25)\\ &= \hat{D}_\rho(1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{\phi}'(\hat{x}) + \epsilon_{\mu\nu}[\hat{M}^{\mu\nu}, \hat{D}_\rho]\hat{\phi}'(\hat{x})\\ &= \hat{D}_\rho\hat{\phi}(\hat{x}) + \epsilon_{\mu\nu}(\eta^\nu{}_\rho\hat{D}^\mu - \eta^\mu{}_\rho\hat{D}^\nu)\hat{\phi}(\hat{x}).\end{aligned}$$

We have used (23) to obtain this result and the fact that the second part on the right-hand side is already ϵ-linear.

The transformation law of a derivative of a scalar field is used to define the transformation law of a vector field:

$$(1 + \epsilon_{\mu\nu}\hat{M}^{\mu\nu})\hat{V}'_\rho(\hat{x}) = \hat{V}_\rho(\hat{x}) + \epsilon_{\mu\nu}(\eta^\nu{}_\rho\hat{V}^\mu - \eta^\mu{}_\rho\hat{V}^\nu). \quad (26)$$

Thus, the derivative

$$\hat{\mathcal{D}}_\rho = \hat{D}_\rho - \mathrm{i}\hat{V}_\rho \quad (27)$$

is κ-Lorentz covariant.

Gauge covariant derivatives

Gauge theories are based on a gauge group. This is a compact Lie group with generators T^a:

$$[T^a, T^b] = \mathrm{i}f^{ab}{}_c T^c. \quad (28)$$

Fields are supposed to span linear representations of this group. Infinitesimal transformations with constant parameters α_a are

$$\delta_\alpha\psi = \mathrm{i}\alpha\psi, \qquad \alpha := \sum_a \alpha_a T^a = \alpha_a T^a. \quad (29)$$

As usual, α is Lie algebra-valued and the commutator of two such transformations closes into a Lie algebra-valued transformation:

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi = [\alpha, \beta]\psi = \mathrm{i}\alpha_a\beta_b f^{ab}{}_c T^c\psi \equiv \delta_{\alpha\times\beta}\psi. \quad (30)$$

The symbol $\alpha \times \beta$ is defined by this equation and it is independent of the representation of the generators T^a.

We generalize the gauge transformations (29) by considering $\hat{x}$-dependent parameters $\hat{\alpha}_a(\hat{x})$. Whereas for commuting coordinates Lie algebra-valued transformations close in the Lie algebra, this will not be true for non-commuting coordinates. This effect of non-commutativity leads to a generalization of Lie algebra-valued gauge transformations [8, 9].

There are exactly two representation-independent concepts based on the commutation relations (28). These are the Lie algebra and the enveloping algebra. The enveloping algebra of the Lie algebra is defined as the free algebra generated by the elements T^a and then divided by the ideal generated by the commutation relations (28). It is infinite-dimensional and consists of all the (abstract) products of the generators modulo the relations (28)[2]. Two elements of the enveloping algebra are identified if they can be transformed into each other by the use of the commutation relations (28).

A basis can be chosen for the enveloping algebra; we use the symmetrized products as such a basis and denote elements of the basis with colons:

$$\begin{aligned}:T^a: &= T^a,\\ :T^aT^b: &= \frac{1}{2}(T^aT^b + T^bT^a), \quad (31)\\ :T^{a_1}\dots T^{a_l}: &= \frac{1}{l!}\sum_{\sigma\in S_l}(T^{\sigma(a_1)}\dots T^{\sigma(a_l)}).\end{aligned}$$

Any formal product of the generators can be expressed in the above basis by using the commutation relations (28), e.g.

$$T^aT^b = \frac{1}{2}\{T^a, T^b\} + \frac{1}{2}[T^a, T^b] =: T^aT^b : + \frac{\mathrm{i}}{2}f^{ab}{}_c : T^c : . \quad (32)$$

The new concept is to allow gauge transformations that are enveloping algebra-valued:

$$\begin{aligned}\hat{\Lambda}_{\{\alpha\}}(\hat{x}) &= \sum_{l=1}^{\infty}\sum_{\text{basis}} \alpha^l_{a_1\dots a_l}(\hat{x}) : T^{a_1}\dots T^{a_l} : \quad (33)\\ &= \alpha_a(\hat{x}) : T^a : + \alpha^2_{a_1a_2} : T^{a_1}T^{a_2} : + \dots\end{aligned}$$

[2] Note that the product is not the matrix product of the generators in a particular representation.

With this definition we write this generalized transformation law as follows:

$$\delta_{\{\alpha\}}\hat{\psi}(\hat{x}) = \mathrm{i}\hat{\Lambda}_{\{\alpha\}}(\hat{x})\hat{\psi}(\hat{x}), \tag{34}$$

where $\{\alpha\}$ denotes the set of the coefficient functions. It is clear that the commutator of two enveloping algebra-valued transformations will be enveloping algebra-valued again. The price we have to pay are the infinitely many parameters $\{\alpha\}$. This is too expensive. But in the next chapter we will get a price reduction. We will find in the next section that we may define the enveloping algebra-valued transformation such that there will only be as many independent parameters as there are in the Lie algebra-valued case. Therefore it is worthwhile to pursue this idea.

It can be seen that under these generalized gauge transformations a covariant derivative

$$\delta_{\{\alpha\}}\left(\hat{\mathcal{D}}_\mu\hat{\psi}(\hat{x})\right) = \mathrm{i}\hat{\Lambda}_{\{\alpha\}}(\hat{x})\hat{\mathcal{D}}_\mu\hat{\psi}(\hat{x}) \tag{35}$$

has to become enveloping algebra-valued as well, by adding an enveloping algebra-valued gauge field:

$$\hat{\mathcal{D}}_\mu = \hat{D}_\mu - \mathrm{i}\hat{V}_\mu,$$
$$\hat{V}_\mu = \sum_{l=1}^{\infty}\sum_{\text{basis}} V^l_{\mu a_1\ldots a_l} : T^{a_1}\ldots T^{a_l} : . \tag{36}$$

Comparing with (27), the gauge field $\hat{V}_\mu$ has to be a vector field under κ-Lorentz transformations. Therefore each gauge field $\hat{V}^l_{\mu\, a_1\ldots a_l}$ has to transform vector-like to guarantee (26).

A new feature arises due to the deformed coproducts (13) of the Dirac operator which we used to define the covariant derivative. We write (35) more explicitly:

$$\begin{aligned}&\delta_{\{\alpha\}}\left((\hat{D}_\mu - \mathrm{i}\hat{V}_\mu)\hat{\psi}(\hat{x})\right)\\ &= \mathrm{i}\hat{D}_\mu\left(\hat{\Lambda}_{\{\alpha\}}(\hat{x})\hat{\psi}(\hat{x})\right) + \hat{V}_\mu\hat{\Lambda}_{\{\alpha\}}\hat{\psi}(\hat{x})\\ &\qquad -\mathrm{i}\left(\delta_{\{\alpha\}}\hat{V}_\mu\right)\hat{\psi}(\hat{x})\\ &\stackrel{!}{=} \mathrm{i}\hat{\Lambda}_{\{\alpha\}}(\hat{x})(\hat{D}_\mu - \mathrm{i}\hat{V}_\mu)\hat{\psi}(\hat{x}).\end{aligned} \tag{37}$$

Both $\hat{D}_n$ and $\hat{D}_i$ act in a non-trivial way on products of functions. For example, (35) will be satisfied for $\hat{\mathcal{D}}_i$ if

$$(\delta_{\{\alpha\}}\hat{V}_i)\hat{\psi} = (\hat{D}_i\hat{\Lambda}_{\{\alpha\}})\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\hat{\psi} - \mathrm{i}[\hat{V}_i, \hat{\Lambda}_{\{\alpha\}}]\hat{\psi}. \tag{38}$$

If we want to use this formula in such a way that it is independent of $\hat{\psi}$, we see from (38) that the gauge field has to be derivative-valued. Only then the transformation

$$\delta_{\{\alpha\}}\hat{V}_i = (\hat{D}_i\hat{\Lambda}_{\{\alpha\}})\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} - \mathrm{i}[\hat{V}_i, \hat{\Lambda}_{\{\alpha\}}], \tag{39}$$

will lead to (35). For $\hat{V}_n$ we can proceed in the same way and find

$$\begin{aligned}\delta_{\{\alpha\}}\hat{V}_n &= (\hat{D}_n\hat{\Lambda}_{\{\alpha\}})\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} + \left((\mathrm{e}^{\mathrm{i}a\hat{\partial}_n} - 1)\hat{\Lambda}_{\{\alpha\}}\right)\hat{D}_n\\ &\quad -\mathrm{i}a(\hat{D}_j\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{\Lambda}_{\{\alpha\}})\hat{D}^j - \mathrm{i}[\hat{V}_n, \hat{\Lambda}_{\{\alpha\}}].\end{aligned} \tag{40}$$

The gauge fields have to be derivative-valued to accommodate the first three terms on the right-hand side of (40) (first term on the right-hand side of (39)). That the gauge fields appear as derivative-valued is a new feature and is a direct consequence of the coproduct rules. We will discuss more details in the next section.

The commutator of two covariant derivatives defines a covariantly transforming object:

$$\hat{\mathcal{F}}_{\mu\nu} = \mathrm{i}[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu]. \tag{41}$$

It will be enveloping algebra- and derivative-valued as it is the case for the gauge field. Its transformation properties are tensor-like:

$$\delta_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu}\hat{\psi} = (\delta_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu})\hat{\psi} + \hat{\mathcal{F}}_{\mu\nu}\delta_{\{\alpha\}}\hat{\psi}. \tag{42}$$

From the definition of the covariant derivative follows

$$\begin{aligned}\delta_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu}\hat{\psi} &= \mathrm{i}\hat{\Lambda}_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu}\hat{\psi}\\ &= \mathrm{i}(\hat{\Lambda}_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu} - \hat{\mathcal{F}}_{\mu\nu}\hat{\Lambda}_{\{\alpha\}})\hat{\psi} + \mathrm{i}\hat{\mathcal{F}}_{\mu\nu}\hat{\Lambda}_{\{\alpha\}}\hat{\psi}.\end{aligned} \tag{43}$$

Comparing this with (42) and (34) shows the covariant transformation property of $\hat{\mathcal{F}}_{\mu\nu}$:

$$\delta_{\{\alpha\}}\hat{\mathcal{F}}_{\mu\nu} = \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{\mathcal{F}}_{\mu\nu}]. \tag{44}$$

The tensor $\hat{\mathcal{F}}_{\mu\nu}$ is derivative-valued. Instead of expanding it in terms of the derivatives $\hat{\partial}_\mu$, we can expand it in terms of covariant derivatives $\hat{\mathcal{D}}_\mu$.

First we express $\hat{\partial}_n$ by $\hat{D}_\mu$ [6]:

$$\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} = -\mathrm{i}a\hat{D}_n + \sqrt{1 + a^2\hat{D}_\mu\hat{D}^\mu}. \tag{45}$$

Next we replace $\hat{D}_\mu$ by $\hat{\mathcal{D}}_\mu$ and subtract the additional terms introduced that way:

$$\hat{D}_n = \hat{\mathcal{D}}_n + \mathrm{i}\hat{V}_n. \tag{46}$$

Each $\hat{V}_n$ will be derivative-valued again, but each derivative carries a factor a and thus contributes to the next order in a. To first order in a we obtain from (45) and (46):

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} &\to 1 - \mathrm{i}a\hat{\partial}_n = 1 - \mathrm{i}a\hat{D}_n\\ &= 1 - \mathrm{i}a\hat{\mathcal{D}}_n + a\hat{V}_n.\end{aligned}$$

To lowest order in a (compare with (40)), $\hat{V}_n$ is not derivative-valued and contributes to the term in $\hat{\mathcal{F}}_{\mu\nu}$ that has no derivatives. Finally we arrive at the expression

$$\hat{\mathcal{F}}_{\mu\nu} = \hat{F}_{\mu\nu} + \hat{T}^{\rho}_{\mu\nu}\hat{\mathcal{D}}_\rho + \ldots + \hat{T}^{\rho_1\ldots\rho_l}_{\mu\nu} : \hat{\mathcal{D}}_{\rho_1}\ldots\hat{\mathcal{D}}_{\rho_l} : + \ldots \tag{47}$$

The colons denote a basis in the free algebra of covariant derivatives. To each finite order in a this expansion will have a finite number of terms. The individual terms will transform like tensors as well:

$$\hat{\mathcal{F}}_{\mu\nu} \to \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{\mathcal{F}}_{\mu\nu}]$$

$$= \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{F}_{\mu\nu}] + \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{T}^{\rho}_{\mu\nu}\hat{\mathcal{D}}_\rho] + \ldots + \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{T}^{\rho_1\ldots\rho_l}_{\mu\nu} : \hat{\mathcal{D}}_{\rho_1}\ldots\hat{\mathcal{D}}_{\rho_l} :] + \ldots \quad (48)$$

When we apply this to a field $\hat{\psi}$ we find as before in (42) to (44)

$$\delta_{\{\alpha\}}\hat{F}_{\mu\nu} = \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{F}_{\mu\nu}], \quad (49)$$

$$\delta_{\{\alpha\}}\hat{T}^{\rho_1\ldots\rho_l}_{\mu\nu} = \mathrm{i}[\hat{\Lambda}_{\{\alpha\}}, \hat{T}^{\rho_1\ldots\rho_l}_{\mu\nu}]. \quad (50)$$

Thus, $\hat{\mathcal{F}}_{\mu\nu}$ can be expanded in terms of a full series of derivative-independent tensors. For the dynamics (Lagrangian) we are only going to use the curvature-like term $\hat{F}_{\mu\nu}$. It transforms like a tensor and reduces to the usual field strength $F^0_{\mu\nu}$ for $a \to 0$. To first order we get one torsion-like contribution $\hat{T}^{\rho}_{\mu\nu}$.

4 Seiberg–Witten map

In the previous chapter we saw that an enveloping algebra-valued gauge transformation depends on an infinite set of parameters. The same is true for the enveloping algebra-valued gauge field; it depends on an infinite set of vector fields. This gauge theory would feature an infinite number of independent degrees of freedom.

This unphysical situation can be avoided if we make the additional assumption that the transformation parameters $\Lambda_{\{\alpha\}}$ depend on the usual, Lie algebra-valued gauge field $A^0_{\mu a}$ [7,8]. We will find that this dependence reduces the infinite number of degrees of freedom of the deformed gauge theory to the finite number of degrees of freedom of the Lie algebra gauge theory.

To find this dependence, known as the Seiberg–Witten map, we start from the gauge transformation:

$$\delta_{\{\alpha\}}\hat{\psi} = \mathrm{i}\hat{\Lambda}_{\{\alpha\}}\hat{\psi}. \quad (51)$$

The condition that this is actually a gauge transformation reads

$$(\delta_{\{\alpha\}}\delta_{\{\beta\}} - \delta_{\{\beta\}}\delta_{\{\alpha\}})\hat{\psi} = \delta_{\{\alpha\times\beta\}}\hat{\psi}. \quad (52)$$

We now introduce Λ_α as opposed to $\Lambda_{\{\alpha\}}$ referring to solutions of the Seiberg–Witten map. We have to replace all parameters in (33) by

$$\alpha^l_{a_1\ldots a_l}(\hat{x}) \longrightarrow \alpha^l_{a_1\ldots a_l}(x;\alpha_a(x), A^0_{\mu a}(x)). \quad (53)$$

The parameters are functions of x, the parameters $\alpha_a(x) \equiv \alpha^1_a(x)$ and the gauge field $A^0_{\mu a}(x)$ as well as of their derivatives. Since we define that the non-commutative gauge parameters have a functional dependence only on commuting variables, we have to use the $\star$- product formalism. We choose as a starting point [10]

$$\delta_\alpha\psi = \mathrm{i}\Lambda_\alpha \star \psi \quad \text{with} \quad (\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi = \delta_{\alpha\times\beta}\psi. \quad (54)$$

The Lie algebra-valued gauge field A^0_μ (in the following we omit all explicit dependence on coordinates x):

$$A^0_\mu = A^0_\mu(x) = \sum_a A^0_{\mu a}(x)T^a \quad (55)$$

has the transformation property

$$\delta_\alpha A^0_\mu = \partial_\mu\alpha - \mathrm{i}[A^0_\mu, \alpha], \quad (56)$$

where $\alpha = \alpha_a(x)T^a$ is Lie algebra-valued as well.

The gauge parameter Λ_α depends on A^0_μ and because of (56) $\delta_\alpha\Lambda_\beta$ is not zero. We take this into account when we write (54) more explicitly

$$\begin{aligned}&(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi \\ &= (\Lambda_\alpha\star\Lambda_\beta - \Lambda_\beta\star\Lambda_\alpha)\star\psi + \mathrm{i}(\delta_\alpha\Lambda_\beta - \delta_\beta\Lambda_\alpha)\star\psi \\ &= \delta_{\alpha\times\beta}\psi.\end{aligned} \quad (57)$$

That (57) has a solution can be seen on more general grounds [12] (also [13–15]). Here we construct a solution by a power series expansion in the deformation parameter a:

$$\Lambda_\alpha = \alpha + a\Lambda^1_\alpha + \ldots + a^k\Lambda^k_\alpha + \ldots\ . \quad (58)$$

In this paper we will consider only the first-order term in a to make the formalism transparent. Assuming a to be small, only the leading term will be of relevance for phenomenological applications. We have however calculated all quantities also to second order and have checked the validity of the statements made here.

We expand (57) to first order in a:

$$\begin{aligned}&\Lambda^0_\alpha\Lambda^1_\beta + \Lambda^1_\alpha\Lambda^0_\beta + \Lambda^0_\alpha\star\Lambda^0_\beta|_{\mathcal{O}(a)} - \Lambda^0_\beta\Lambda^1_\alpha \\ &- \Lambda^1_\beta\Lambda^0_\alpha - \Lambda^0_\beta\star\Lambda^0_\alpha|_{\mathcal{O}(a)} + \mathrm{i}(\delta_\alpha\Lambda^1_\beta - \delta_\beta\Lambda^1_\alpha) = \mathrm{i}\Lambda^1_{\alpha\times\beta},\end{aligned} \quad (59)$$

or, using $\Lambda^0_\alpha = \alpha, \Lambda^0_\beta = \beta$ and the explicit form of the $\star$-product,

$$\begin{aligned}&[\alpha, \Lambda^1_\beta] + [\Lambda^1_\alpha, \beta] + \frac{\mathrm{i}}{2}x^\lambda C^{\mu\nu}_\lambda\{\partial_\mu\alpha, \partial_\nu\ \beta\} \\ &+\mathrm{i}(\delta_\alpha\Lambda^1_\beta - \delta_\beta\Lambda^1_\alpha) = \mathrm{i}\Lambda^1_{\alpha\times\beta}\,.\end{aligned} \quad (60)$$

To first order in a the non-commutative structure contributes a term from the $\star$-product, which forbids Λ^1_α equal zero. Equation (60) is an inhomogeneous linear equation in Λ^1_α, with the solution

$$\Lambda^1_\alpha = -\frac{1}{4}x^\lambda C^{\mu\nu}_\lambda\{A^0_\mu, \partial_\nu\ \alpha\}, \quad (61)$$

where $C^{\mu\nu}_\lambda$ are the structure constants of the coordinate algebra. More explicitly this is

$$\Lambda^1_\alpha = \frac{a}{4}x^j\left(\{A^0_j, \partial_n\alpha\} - \{A^0_n, \partial_j\alpha\}\right). \quad (62)$$

This solution is hermitean for real fields $A^0_{\mu a}(x)$ and real parameters $\alpha_a(x)$. That this specific solution of the inhomogeneous equation is not unique and that it is possible to add to it solutions of the homogeneous equation

$$[\alpha, \Lambda^1_\beta] + [\Lambda^1_\alpha, \beta] + \mathrm{i}(\delta_\alpha\Lambda^1_\beta - \delta_\beta\Lambda^1_\alpha) = \mathrm{i}\Lambda^1_{\alpha\times\beta} \quad (63)$$

has been discussed thoroughly in many places (e.g. [10,15]). There are no new aspects to this question in first order in a in this particular model.

With a solution for Λ^1_α at our disposition, it is possible to express a "matter" field ψ (i.e. field in the fundamental representation) in terms of A^0_μ and a matter field ψ^0 of the standard gauge theory

$$\delta_\alpha \psi^0 = \mathrm{i}\alpha\psi^0. \tag{64}$$

Up to first order in a, (54) is solved by

$$\psi = \psi^0 - \frac{1}{2}x^\lambda C^{\mu\nu}_\lambda A^0_\mu \partial_\nu \psi^0 + \frac{\mathrm{i}}{8}x^\lambda C^{\mu\nu}_\lambda [A^0_\mu, A^0_\nu]\psi^0. \tag{65}$$

The same way as we express ψ in terms of A^0_μ and ψ^0, we can define the Seiberg–Witten map for gauge fields (they are in the adjoint representation of the enveloping algebra). When we derived the respective formulae in the previous section, we discovered that the gauge fields have to be derivative-valued. Therefore we have to discuss solutions of the Seiberg–Witten map for the following relations:

$$\delta_\alpha V_i = (D^*_i \Lambda_\alpha)\mathrm{e}^{-\mathrm{i}a\partial_n} - \mathrm{i}[V_i \stackrel{\star}{,} \Lambda_\alpha], \tag{66}$$

and

$$\begin{aligned}\delta_\alpha V_n = (D^*_n \Lambda_\alpha)\mathrm{e}^{-\mathrm{i}a\partial_n} + \left((\mathrm{e}^{\mathrm{i}a\partial_n} - 1)\Lambda_\alpha\right) D^*_n \\ -\mathrm{i}a(D^*_j \mathrm{e}^{\mathrm{i}a\partial_n}\Lambda_\alpha)D^{*j} - \mathrm{i}[V_n \stackrel{\star}{,} \Lambda_\alpha].\end{aligned} \tag{67}$$

It is technically not simple to solve these two equations (especially since the second one is a sum of several terms with different dependence on derivatives), but conceptually there are no further problems. Without going into details we present the solution up to first order in a:

$$\begin{aligned}V_i &= A^0_i - \mathrm{i}aA^0_i\partial_n - \frac{\mathrm{i}a}{2}\partial_n A^0_i - \frac{a}{4}\{A^0_n, A^0_i\} \\ &\quad + \frac{1}{4}x^\lambda C^{\mu\nu}_\lambda \left(\{F^0_{\mu i}, A^0_\nu\} - \{A^0_\mu, \partial_\nu A^0_i\}\right),\end{aligned} \tag{68}$$

$$\begin{aligned}V_n &= A^0_n - \mathrm{i}aA^{0j}\partial_j - \frac{\mathrm{i}a}{2}\partial_j A^{0j} - \frac{a}{2}A^0_j A^{0j} \\ &\quad + \frac{1}{4}x^\lambda C^{\mu\nu}_\lambda \left(\{F^0_{\mu n}, A^0_\nu\} - \{A^0_\mu, \partial_\nu A^0_n\}\right).\end{aligned} \tag{69}$$

Here $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu - \mathrm{i}[A^0_\mu, A^0_\nu]$ is the field strength of the undeformed gauge theory. We emphasize the dependence on derivatives in the terms $A^0_i\partial_n$ and $A^{0j}\partial_j$.

From the covariant derivative $\mathcal{D}_\mu = D^*_\mu - \mathrm{i}V_\mu$ we can calculate $\mathcal{F}_{\mu\nu} = \mathrm{i}[\mathcal{D}_\mu \stackrel{\star}{,} \mathcal{D}_\nu]$ to first order in a. As discussed in the previous section, it will be of first order in the derivatives, the sum of a curvature-like term and a torsion-like term:

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + T^\rho_{\mu\nu}\mathcal{D}_\rho. \tag{70}$$

The result is (up to first order in a)

$$\begin{aligned}F_{ij} &= F^0_{ij} - \mathrm{i}a\mathcal{D}^0_n F^0_{ij} \\ &\quad + \frac{1}{4}x^\lambda C^{\mu\nu}_\lambda \left(2\{F^0_{\mu i}, F^0_{\nu j}\} + \{\mathcal{D}^0_\mu F^0_{ij}, A^0_\nu\} \right. \\ &\quad \left. -\{A^0_\mu, \partial_\nu F^0_{ij}\}\right),\end{aligned} \tag{71}$$

$$T^\mu_{ij} = -2\mathrm{i}a\eta^\mu{}_n F^0_{ij}, \tag{72}$$

$$\begin{aligned}F_{nj} &= F^0_{nj} - \frac{\mathrm{i}a}{2}\mathcal{D}^{\mu 0}F^0_{\mu j} \\ &\quad + \frac{1}{4}x^\lambda C^{\mu\nu}_\lambda \left(2\{F^0_{\mu n}, F^0_{\nu j}\} + \{\mathcal{D}^0_\mu F^0_{nj}, A^0_\nu\} \right. \\ &\quad \left. - \{A^0_\mu, \partial_\nu F^0_{nj}\}\right),\end{aligned} \tag{73}$$

$$T^\mu_{nj} = -\mathrm{i}a\eta^{\mu l}F^0_{lj} - \mathrm{i}a\eta^\mu{}_n F^0_{nj}. \tag{74}$$

These quantities transform covariantly:

$$\delta_\alpha F_{\mu\nu} = \mathrm{i}[\Lambda_\alpha \stackrel{\star}{,} F_{\mu\nu}], \tag{75}$$

$$\delta_\alpha T^\rho_{\mu\nu} = \mathrm{i}[\Lambda_\alpha \stackrel{\star}{,} T^\rho_{\mu\nu}]. \tag{76}$$

Now we have all the ingredients to construct to first order in a a gauge theory based on the non-commutative spaces defined by (1) in terms of the usual fields A^0_μ and ψ^0.

The dynamics of the gauge field can be formulated with the tensor $F^{\mu\nu}$

$$\mathcal{L}_{\text{gauge}} = c\ \mathrm{Tr}\left(F^{\mu\nu} \star F_{\mu\nu}\right). \tag{77}$$

Note, however, that $\mathrm{Tr}\left(F_{\mu\nu} \star F^{\mu\nu}\right)$ is not invariant because the coordinates do not commute. The Lagrangian $\mathcal{L}_{\text{gauge}}$ will render an action gauge invariant if it is formulated with an integral with the trace property[3]. The trace will also depend on the representation of the generators T^a because higher products of the generators will enter through the enveloping algebra (for a detailed discussion of this issue, see [17]).

To first order in a, when written in terms of A^0_μ, we obtain the following expression for the gauge part of the Lagrangian (choosing in analogy to the undeformed theory $c = -\frac{1}{4}$):

$$\begin{aligned}&\mathcal{L}_{\text{gauge}}|_{\mathcal{O}(a)} = -\frac{\mathrm{i}}{8}x^\lambda C^{\rho\sigma}_\lambda \\ &\times \mathrm{Tr}\left(\mathcal{D}^0_\rho F^{0\mu\nu}\mathcal{D}^0_\sigma F^0_{\mu\nu} + \frac{\mathrm{i}}{2}\{A^0_\rho, (\partial_\sigma + \mathcal{D}^0_\sigma)(F^{0\mu\nu}F^0_{\mu\nu})\}\right. \\ &\quad \left. -\mathrm{i}\{F^{0\mu\nu}, \{F^0_{\mu\rho}, F^0_{\nu\sigma}\}\}\right) \\ &\quad + \frac{\mathrm{i}a}{4}\mathrm{Tr}\left(\mathcal{D}^0_n(F^{0\mu\nu}F^0_{\mu\nu}) - \{\mathcal{D}^0_\mu F^{0\mu j}, F^0_{nj}\}\right),\end{aligned} \tag{78}$$

where $\mathcal{D}^0_\mu = \partial_\mu - \mathrm{i}A^0_\mu$ (or adjoint $\mathcal{D}^0_\mu \cdot = \partial_\mu \cdot - \mathrm{i}[A^0_\mu, \cdot]$ acting on $F^0_{\mu\nu}$). Cyclicity of the trace allows for several simplifications on the terms on the right-hand side.

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[3] To attain the trace property, a measure function can be introduced (compare [6]). Since the measure function does in general not vanish in the limit $a \to 0$, it should be compensated without spoiling the gauge invariance of the action. This is possible, leading however to additional first-order terms in the action (compare e.g. [16]).

The matter part of the theory will be the gauge covariant version of the free Lagrangian as it was developed in [6]

$$\mathcal{L}_{\text{matter}} = \bar{\psi} \star (\mathrm{i}\gamma^\mu \mathcal{D}_\mu - m)\,\psi. \tag{79}$$

To first order in a, when written in terms of A^0_μ and ψ^0, we obtain

$$\begin{aligned}\mathcal{L}_{\text{matter}}|_{\mathcal{O}(a)} = &\frac{\mathrm{i}}{2} x^\nu C^{\rho\sigma}_\nu \overline{\mathcal{D}^0_\rho \psi^0}\ \mathcal{D}^0_\sigma \left(\mathrm{i}\gamma^\mu \mathcal{D}^0_\mu - m\right)\psi^0 \\ &- \frac{\mathrm{i}}{2} x^\nu C^{\rho\sigma}_\nu \bar{\psi}^0 \gamma^\mu F^0_{\mu\rho} \mathcal{D}^0_\sigma \psi^0 \\ &+ \frac{a}{2} \bar{\psi}^0 \gamma^j \mathcal{D}^0_n \mathcal{D}^0_j \psi^0 + \frac{a}{2} \bar{\psi}^0 \gamma^n \mathcal{D}^0_j \mathcal{D}^{0j} \psi^0.\end{aligned} \tag{80}$$

These are the Lagrangians which define the dynamics on the κ-deformed Minkowski space.

5 Gauge transformations and the κ-Lorentz algebra

Our concept of gauge transformations on non-commutative spaces rests on the Seiberg–Witten map. With the help of this map gauge transformations can be realized in the enveloping algebra of the Lie algebra

$$\begin{aligned}\phi' = \phi + \delta_\alpha \phi = \phi + \mathrm{i}\Lambda_\alpha \star \phi \\ \text{with} \quad (\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\phi = \delta_{\alpha\times\beta}\phi.\end{aligned} \tag{81}$$

To find such a realization it turned out to be necessary that Λ_α depends on the standard Lie algebra-valued gauge field A^0_μ and its derivatives. Therefore under a gauge transformation Λ_α will transform as well and (81) leads to

$$\begin{aligned}&\mathrm{i}(\delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha) \star \phi + (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \phi \\ &\quad = \mathrm{i}\Lambda_{\alpha\times\beta} \star \phi.\end{aligned} \tag{82}$$

In this section we want to see how these equations behave under the κ-deformed Lorentz transformations. Only M^{*in} has a deformed coproduct rule (compare with (21)):

$$\begin{aligned}&M^{*in}(f \star g) \\ &= \left(M^{*in} f\right) \star g + \left(\mathrm{e}^{\mathrm{i}a\partial^*_n} f\right) \star \left(M^{*in} g\right) \\ &\qquad + \mathrm{i}a \left(\partial^*_j f\right) \star \left(M^{*ij} g\right)\end{aligned} \tag{83}$$

and therefore we will restrict our discussion to $N_\epsilon = \epsilon_i M^{in}$.

A scalar field transforms as follows:

$$\tilde{\phi} = \phi - N^*_\epsilon \phi, \tag{84}$$

where N^*_ϵ acts on the coordinates; compare with (23). This transformation can be inverted, to first order in ϵ:

$$\phi = \tilde{\phi} + N^*_\epsilon \phi. \tag{85}$$

We assume that Λ_α transforms like a scalar field.

First we compute $\widetilde{\phi'}$, by applying (84) to (81):

$$\widetilde{\phi'} = \phi + \mathrm{i}\Lambda_\alpha \star \phi - (N^*_\epsilon \phi) - \mathrm{i}N^*_\epsilon(\Lambda_\alpha \star \phi). \tag{86}$$

For evaluating the last term in (86), the coproduct (83) has to be used.

Next we compute $\tilde{\phi}\,'$ by applying (81) to (84):

$$\tilde{\phi}\,' = \phi - (N^*_\epsilon \phi) + \mathrm{i}\Lambda_\alpha \star \phi - \mathrm{i}N^*_\epsilon(\Lambda_\alpha \star \phi). \tag{87}$$

This shows that the two transformations commute. When we use (85), the gauge transformation (86) can be written as a gauge transformation on $\tilde{\phi}$:

$$\delta_\alpha \tilde{\phi} = \mathrm{i}\Lambda_\alpha \star \tilde{\phi} + \mathrm{i}\Lambda_\alpha \star (N^*_\epsilon \tilde{\phi}) - \mathrm{i}N^*_\epsilon(\Lambda_\alpha \star \tilde{\phi}). \tag{88}$$

We draw the commuting diagram to illustrate the result

$$\begin{array}{ccc}\phi & \xrightarrow{\alpha} & \phi' \\ \epsilon\downarrow & & \downarrow\epsilon \\ \tilde{\phi} & \xrightarrow{\alpha} & \widetilde{\phi'} \equiv \tilde{\phi}\,'.\end{array} \tag{89}$$

The gauge transformations on $\tilde{\phi}$ – the κ-Lorentz transformed scalar field – is now defined by (88). It remains to be shown that (88) realizes the gauge group as well:

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\tilde{\phi} = \delta_{\alpha\times\beta}\tilde{\phi}. \tag{90}$$

It is easier to compute $\delta_\beta\delta_\alpha\tilde{\phi}$ from (87) and to use (81). We make use of (83) and after some rearrangements we obtain

$$\begin{aligned}&(\delta_\beta\delta_\alpha - \delta_\alpha\delta_\beta)\tilde{\phi} \\ &= (\mathrm{i}(\delta_\beta\Lambda_\alpha - \delta_\alpha\Lambda_\beta) - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)) \star \phi \\ &\quad - N^*_\epsilon\,(\mathrm{i}(\delta_\beta\Lambda_\alpha - \delta_\alpha\Lambda_\beta) \\ &\qquad - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)) \star \phi \\ &\quad - \mathrm{e}^{\mathrm{i}a\partial^*_n}\,(\mathrm{i}(\delta_\beta\Lambda_\alpha - \delta_\alpha\Lambda_\beta) \\ &\qquad - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)) \star N^*_\epsilon \phi \\ &\quad + \mathrm{i}a\partial^*_j\,(\mathrm{i}(\delta_\beta\Lambda_\alpha - \delta_\alpha\Lambda_\beta) \\ &\qquad - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)) \star \epsilon_l M^{lj} \phi.\end{aligned} \tag{91}$$

We use the condition (82) again and obtain the result (90). This demonstrates that (88) is a gauge transformation.

It is also possible to verify the result (90) by a direct calculation. We start with the solution of the Seiberg–Witten map (62)

$$\begin{aligned}\Lambda_\alpha &= \alpha - \frac{1}{4} x^\lambda C^{\mu\nu}_\lambda \{A^0_\mu, \partial_\nu\,\alpha\} + \mathcal{O}(a^2) \\ &=: \alpha + \Lambda^1_\alpha + \mathcal{O}(a^2)\end{aligned} \tag{92}$$

and (65)

$$\begin{aligned}\phi &= \phi^0 - \frac{1}{2}x^\mu C_\mu^{\rho\sigma} A_\rho^0 \partial_\sigma \phi^0 \\ &+ \frac{\mathrm{i}}{8}x^\mu C_\mu^{\rho\sigma}[A_\rho^0, A_\sigma^0]\phi^0 + \mathcal{O}(a^2) \\ &=: \phi^0 + \phi^1 + \mathcal{O}(a^2)\,. \end{aligned}\tag{93}$$

We first apply M^{*in} to (93) and gauge transform the undeformed fields afterwards. This has to be equal to M^{*in} applied to $\delta_\alpha\phi = \mathrm{i}\Lambda_\alpha \star \phi$ up to first order in a. Applying M^{*in} on $\delta_\alpha\phi$, the coproduct (83) has to be taken into account and we obtain

$$\begin{aligned}&M^{*in}(\Lambda_\alpha \star \phi) \\ &= (M^{*in}\Lambda_\alpha)\star\phi + (\mathrm{e}^{\mathrm{i}a\partial_n^*}\Lambda_\alpha)\star(M^{*in}\phi) \\ &\quad +\mathrm{i}a(\partial_j^*\Lambda_\alpha)\star(M^{ij*}\phi).\end{aligned}\tag{94}$$

To write this explicitly to first order we need the operators (21) expanded up to first order in a:

$$\begin{aligned}M^{*in} &= x^i\partial^n - x^n\partial^i + \frac{\mathrm{i}a}{2}x^i\partial_\mu\partial^\mu - \frac{\mathrm{i}a}{2}x^\mu\partial_\mu\partial^i \\ &=: M_0^{*in} + M_1^{*in}\end{aligned}\tag{95}$$

and

$$M^{ij*} = x^i\partial^j - x^j\partial^i =: M_0^{ij*}\,.\tag{96}$$

Now we obtain from (94):

$$\begin{aligned}&\mathrm{i}M^{*in}(\Lambda_\alpha\star\phi)|_{\mathcal{O}(a)} \\ &= \mathrm{i}(M_1^{*in}\alpha)\phi^0 + \mathrm{i}\alpha(M_1^{*in}\phi^0) - \frac{1}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho(M_0^{*in}\alpha)\partial_\sigma\phi^0 \\ &\quad - \frac{1}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma(M_0^{*in}\phi^0) + \mathrm{i}(M_0^{*in}\alpha)\phi^1 \\ &\quad + \mathrm{i}\alpha(M_0^{*in}\phi^1) + \mathrm{i}(M_0^{*in}\Lambda_\alpha^1)\phi^0 + \mathrm{i}\Lambda_\alpha^1(M_0^{*in}\phi^0) \\ &\quad - a\partial_n\alpha(M_0^{*in}\phi^0) - a\partial_j\alpha(M_0^{ij*}\phi^0)\,.\end{aligned}\tag{97}$$

Notice that

$$\begin{aligned}&\delta_\alpha(M_0^{*in}\phi^1) \\ &= \mathrm{i}M_0^{*in}\left(\frac{\mathrm{i}}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma\phi^0 + \alpha\phi^1 + \Lambda_\alpha^1\phi^0\right),\end{aligned}\tag{98}$$

since ϕ^1 was constructed as solution for the Seiberg–Witten map (65). Besides it can be shown by direct calculation that

$$\begin{aligned}&\mathrm{i}M_1^{*in}(\alpha)\phi^0 + \mathrm{i}\alpha M_1^{*in}(\phi^0) \\ &= \mathrm{i}M_1^{*in}(\alpha\phi^0) - ax^j\partial_\mu\alpha\partial^\mu\phi^0 + \frac{a}{2}x^\mu\partial_\mu\alpha\partial^u\phi^0 \\ &\quad + \frac{a}{2}x^\mu\partial^i\alpha\partial_\mu\phi^0\end{aligned}\tag{99}$$

as well as that

$$\begin{aligned}&- \frac{1}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho(M_0^{*in}\alpha)\partial_\sigma\phi^0 - \frac{1}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma(M_0^{*in}\phi^0) \\ &= -M_0^{*in}\left(\frac{1}{2}x^\mu C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma\phi^0\right) + \frac{1}{2}M_0^{*in}(x^\mu)C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma\phi^0 \\ &\quad - \frac{1}{2}x^\mu C_\mu^{\rho\sigma}\left(\partial_\rho(M_0^{*in})(\alpha)\partial_\sigma\phi^0 + \partial_\rho\alpha\partial_\sigma(M_0^{*in}\phi^0)\right),\end{aligned}\tag{100}$$

where $\partial_\rho(M_0^{*in})(\alpha) := \eta_\rho^i\partial^n\alpha - \eta_\rho^n\partial^i\alpha$. Then (98), (99) and (100) yield

$$\begin{aligned}&M^{*in}(\Lambda_\alpha\star\phi)|_{\mathcal{O}(a)} \\ &= \delta_\alpha(M_1^{*in}\phi^0) + \delta_\alpha(M_0^{*in}\phi^1) \\ &\quad -ax^i\partial_\mu\alpha\partial^\mu\phi^0 + \frac{a}{2}x^\mu\partial_\mu\alpha\partial^i\phi^0 \\ &\quad +\frac{a}{2}x^\mu\partial^i\alpha\partial_\mu\phi^0 + \frac{1}{2}M_0^{*in}(x^\mu)C_\mu^{\rho\sigma}\partial_\rho\alpha\partial_\sigma\phi^0 \\ &\quad -\frac{1}{2}x^\mu C_\mu^{\rho\sigma}\left(\partial_\rho(M_0^{*in})(\alpha)\partial_\sigma\phi^0 + \partial_\rho\alpha\partial_\sigma(M_0^{*in}\phi^0)\right).\end{aligned}\tag{101}$$

Calculation shows that the last terms on the right-hand side all cancel and we end up with

$$(M^{*in}\delta_\alpha\phi)|_{\mathcal{O}(a)} = \delta_\alpha(M_1^{*in}\phi^0) + \delta_\alpha(M_0^{*in}\phi^1)\,.\tag{102}$$

Hence we showed explicitly up to first order in a that (90) is true.

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